

Finite-time Lyapunov exponents of Strange Nonchaotic Attractors

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The probability distribution of finite-time Lyapunov exponents provides an important characterization of dynamical attractors. We study such distributions for strange nonchaotic attractors (SNAs) created through several different mechanisms in quasiperiodically forced nonlinear dynamical systems. Statistical properties of the distributions such as the variance and the skewness also distinguish between SNAs formed by different bifurcation routes.

I. INTRODUCTION

The characterization of attractors in nonlinear dynamical systems is a problem that has seen considerable progress in the past decade [1,2]. Beyond classification as simple or strange, the calculation of fractal dimensions and the description of the measure in term of a multifractal spectrum of singularities has become an important means of describing the structure of dynamical attractors [1]. For hyperbolic attractors there are rigorous results connecting the multifractal structure, through the thermodynamic formalism, with dynamical information as embodied in finite-time Lyapunov exponents [2,3].

Grebogi *et al.* [4] first described dynamical systems wherein the attractors that result are fractal, but the dynamics is not chaotic, in that the largest Lyapunov exponent is not greater than zero. These strange nonchaotic attractors (SNAs) are generic in quasiperiodically forced systems. Subsequently, considerable effort has been directed toward the characterization and study of SNAs [5–11], which have also been observed experimentally [12–14]. A potential use of SNAs is in the area of secure communications, and recent applications [15,16] exploit the ease of synchronization of such systems.

The strange nonchaotic state is only one of the possible dynamical states realized in quasiperiodically driven systems; periodic, quasiperiodic and chaotic attractors can also be obtained as parameters are varied. SNAs are typically found for parameter values very close to the boundaries of the chaotic regions, and the different bifurcation mechanisms through which they are created is a problem of interest. There are a number of routes or scenarios for the creation of SNAs, some of which can be correlated with bifurcations. These include

- (i) the Heagy–Hammel (HH) [8] mechanism involving a collision between a period-doubled torus and its unstable parent,
- (ii) the blowout bifurcation route [11], and
- (iii) intermittency [17], when as a function of driving parameter a chaotic strange attractor disappears and is eventually replaced by a torus through an analogue of the saddle-node bifurcation.

The signature of these bifurcations in terms of the behaviour of the largest Lyapunov exponent has been discussed in detail [18]. The blowout bifurcation mechanism [11] requires that the quasiperiodic torus of a system with an invariant subspace losses its transverse stability as a parameter changes across the transition and lead to the birth of an SNA. In this process the transverse Lyapunov exponent becomes positive while the nontrivial Lyapunov exponent for the whole system remains negative. In the HH mechanism [8], as system parameters are varied, the period-doubled torus gets progressively more wrinkled and collides with a parent unstable torus; this scenario is like an attractor merging crisis [19]. The distinctive signature of the intermittency route to SNA is a sharp change in the Lyapunov exponent which shows large variance and scaling behavior [17] at the bifurcation.

A general mechanism that is frequently observed but for which there is no well-identified bifurcation is the so-called fractalization route [9], whereby a smooth torus gets increasingly wrinkled and transforms into a SNA without any interaction with a nearby unstable periodic orbit (in contrast to HH). This is probably the most common route to SNA in a number of maps and flows [6,7].

The present paper addresses the issue of distinguishing among SNAs formed by different routes through the use of finite-time (or local) Lyapunov exponents. We show that the morphologies of different SNAs differ in crucial ways, particularly for intermittent SNAs [17,18]. This is seen most dramatically in the characteristic distributions of local Lyapunov exponents and the statistical properties of the distributions such as the variance and the skewness.

In Sec. II, we briefly introduce the dynamical systems that are studied here. Results are discussed in Sec III. This is followed by a summary in Sec. IV.

II. DYNAMICAL SYSTEMS

Several quasiperiodically driven systems—both maps and flows [6–11]—have been shown to have SNAs. We consider the quasiperiodically forced logistic map [8] wherein three of the routes to SNAs can be observed. This system is defined by the equations

$$\begin{aligned} x_{n+1} &= \alpha[1 + \epsilon \cos(2\pi\phi_n)] x_n (1 - x_n), \\ \phi_{n+1} &= \phi_n + \omega \pmod{1}, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^1$, $\phi \in \mathbb{S}^1$, $\omega = (\sqrt{5} - 1)/2$ is the irrational driving frequency, and ϵ represents the forcing amplitude.

The largest nontrivial Lyapunov exponent Λ is defined through

$$\lambda_N(x_n) = \frac{1}{N} \sum_{j=1}^N \ln |dx_{n+j}/dx_{n+j-1}|, \quad (2)$$

$$\Lambda = \lim_{N \rightarrow \infty} \lambda_N(x_n), \quad (3)$$

λ_N being the local or N -step Lyapunov exponent. Note that λ_N depends on the initial condition x_n while Λ , of course, does not.

A region of the phase-diagram [17] of the forced logistic map, Eq. (1) in the $\alpha - \epsilon'$ plane (where $\epsilon' = \epsilon/(4/\alpha - 1)$) is shown in Figure 1. The rescaling of the forcing amplitude is done for convenience [17], and the region of Figure 1 corresponds to the period-3 window of the unforced logistic map. The symbols P, S and C correspond to periodic, strange nonchaotic, and chaotic behavior of the system. The dashed line is the locus of the period doubling bifurcation from period-3 to period-6.

Along the left edge separating the periodic window from the region C_1 , there appear to be no SNAs. On the right edge marked C'_1 , fractalized SNAs are obtained. The intermittent SNAs are found in the region marked **I**. Within this window, it was difficult to locate the HH mechanism for the formation of SNAs which is known to operate in this system for a number of different parameter values [8,17].

We also study the blowout bifurcation route to SNA, which occurs in the mapping [11]

$$\begin{aligned} x_{n+1} &= [a \cos(2\pi\phi_n) + b] \sin(2\pi x_n) \\ \phi_{n+1} &= \phi_n + \omega \pmod{1}, \end{aligned} \quad (4)$$

(a and b are parameters and $\omega = (\sqrt{5} - 1)/2$) which is bounded and has no other stable attractors other than the invariant subspace ($x = 0, \phi$). As parameters changes across the critical values, the dynamics of x leads to a strange nonchaotic attractor [11]. Our interest is in contrasting this mechanism for SNA formation which is also accompanied by on-off intermittency, with the intermittent SNA [17].

III. RESULTS

Although λ_N depends on initial conditions, the probability density, defined through

$$P(N, \lambda) d\lambda = \text{Probability that } \lambda_N \text{ lies between } \lambda \text{ and } \lambda + d\lambda, \quad (5)$$

does not. This distribution can be obtained by taking an (infinitely) long, ergodic trajectory, and dividing it in segments of length N , from which the local Lyapunov exponent can be calculated through Eq. (2).

For chaotic motion it has been argued [2,3] that since the local Lyapunov exponents can be treated as independent random fluctuations, the central limit theorem is valid, leading to a normal distribution for λ_N ,

$$P(N, \lambda) \approx \frac{1}{\sqrt{2\pi N G''(\Lambda)}} \exp[-N G''(\Lambda)(\lambda - \Lambda)^2/2] \quad (6)$$

with the function G , the spectrum of effective Lyapunov exponents [3], being appropriately defined [2].

These expectations are not always satisfied since there can be important correlations in the dynamics. We have recently described the characteristic distributions for finite time Lyapunov exponents in low dimensional chaotic systems where there are significant departures from central-limit behaviour [20].

For SNAs there are additional complications. Although Λ is by definition nonpositive, $P(N, \lambda)$ can have a significant contribution from $\lambda > 0$: for some of the time, these systems behave chaotically because of the fractal structure of the attractors [7]. In the limit of large N , the contribution from positive λ decreases and the density collapses to a δ -function, $\lim_{N \rightarrow \infty} P(N, \lambda) \rightarrow \delta(\Lambda - \lambda)$.

Shown in Figs. 2 and 3 are local Lyapunov distributions for the four routes to SNAs (parameters are specified in the caption), for short ($N = 50$) and long but finite ($N = 1000$) times. The fractalized and HH SNAs both show a gradual approach to the normal distribution (a gaussian is fit to the data in Fig. 3), while the blowout SNAs, and more spectacularly, the intermittent SNAs show a distinctive departure from the gaussian distribution.

The intermittent SNA is morphologically and dynamically very different from the other SNAs, and the shape of the characteristic distribution, $P(N, \lambda)$ is a combination of a gaussian and an exponential [20]. In contrast to the other SNAs, the distribution is asymmetric, and the large λ tail decays very slowly.

It appears that the intermittent SNA is in a distinct universality class [20] and a number of quantitative measures can be devised in order to show this distinction. Consider, for example, the fraction of exponents lying above $\lambda = 0$,

$$F_+(N) = \int_0^\infty P(N, \lambda) d\lambda, \quad (7)$$

$F_+(N)$ vs N for the different SNAs are shown in Fig. 4. Except for the intermittent SNA, for which $F_+(N) \sim N^{-\beta}$, this quantity decays exponentially, $F_+(N) \sim \exp(-\gamma N)$, with the exponents β and γ depending strongly on the parameters of the system. We found that the values of the exponents are $\beta = 0.72$ for the intermittent SNA and $\gamma = 0.02, 0.007$ and 0.042 respectively for HH, fractalized, and blowout SNAs. Similarly, other statistical properties of these distributions can be studied. We calculate the first two moments about the arithmetic mean of all distributions and obtain the variance,

$$\sigma^2 = \int_{-\infty}^\infty (\lambda - \Lambda)^2 P(N, \lambda) d\lambda, \quad (8)$$

and the coefficient of skewness, namely

$$\gamma_1 = \int_{-\infty}^\infty (\lambda - \Lambda)^3 P(N, \lambda) d\lambda / (\sigma^3), \quad (9)$$

which are shown in Fig. 5(a) and Fig. 5(b) respectively. Generally for all types of SNAs, the variance of $P(N, \lambda)$ decreases as a power of N , $\sigma^2 \sim 1/N^\delta$ where the exponent δ is different for each SNA. The variance for intermittent SNAs decreases very slowly compared to other SNAs, and our numerical results for the exponents, for the examples shown here are $= 0.97, 1.71, 1.63$, and 1.7 for intermittent, fractalized, HH, and blowout SNAs. The degree of asymmetry in the distribution is quantified by the significantly larger skewness γ_1 (see Fig. 5(b)).

IV. SUMMARY

In the present paper we have studied the dynamical structure of strange nonchaotic attractors formed by different bifurcation mechanisms in quasiperiodically driven systems, by examining the *distribution* of finite-time Lyapunov exponents. Although the Lyapunov exponent is negative on a SNA, over short times, nearby trajectories can separate from one another since the attractor is strange: this corresponds to a local positive Lyapunov exponent. The manner in which this distribution changes as a function of time is characteristic of the attractor, and of the bifurcation routes through which attractors are created: intermittent dynamics leads to very distinctive distributions of local Lyapunov exponents [20].

Our present results further underscore the utility of finite-time Lyapunov exponents in describing the local structure of dynamical attractors [2,3,20] in general. For the case of hyperbolic attractors, the theory connecting these to the invariant measure is well-developed. The present paper is part of a preliminary step towards understanding the connection between an underlying fractal structure and globally nonchaotic dynamics on strange nonchaotic attractors.

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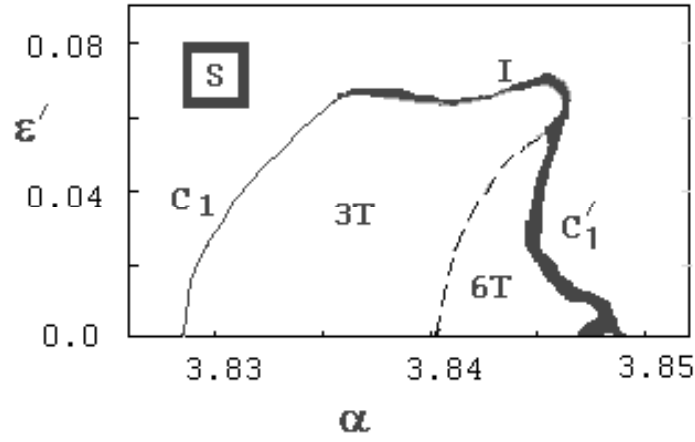


Fig. 1. Phase diagram for the quasiperiodically forced logistic map (schematic)[18], corresponding to the period 3 window of the unforced case. In order to obtain this diagram, Λ is calculated in a 100×100 grid. The shaded region (S) along the boundary (the $\Lambda = 0$ contour) shows the region of SNAs. Intermittent SNAs are found on the edge of the C_1 region marked I, while the right boundary, denoted C_1' has fractalized SNAs. The left boundary between the periodic region and C_1 does not show any SNA. The boundaries of the attractors can be interwoven in complicated manner (especially along C_1').

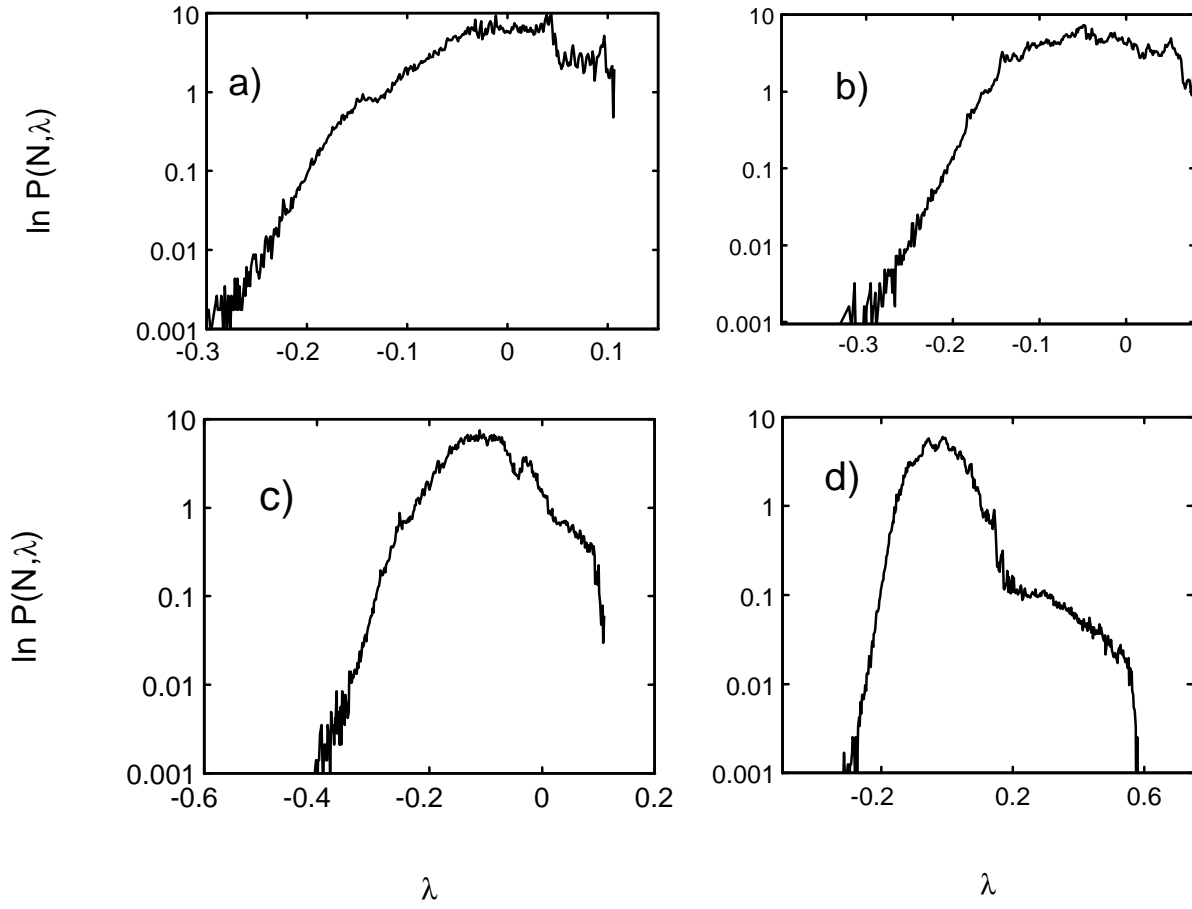


Fig. 2. Density of finite-time Lyapunov exponents $P(50, \lambda)$ for SNAs formed by the four main mechanisms, (a) the fractalized route at $\alpha = 3.84618$ for $\epsilon' = 0.06$ in Eq. (1), (b) the HH route at $\alpha = 3.2745$ for $\epsilon' = 0.5$ in Eq. (1), (c) the blowout bifurcation route at $a = 1$ and $b = 2.1$ in Eq. (4), and (d) the intermittency route at $\alpha = 3.84549$ for $\epsilon' = .073$ in Eq. (1).

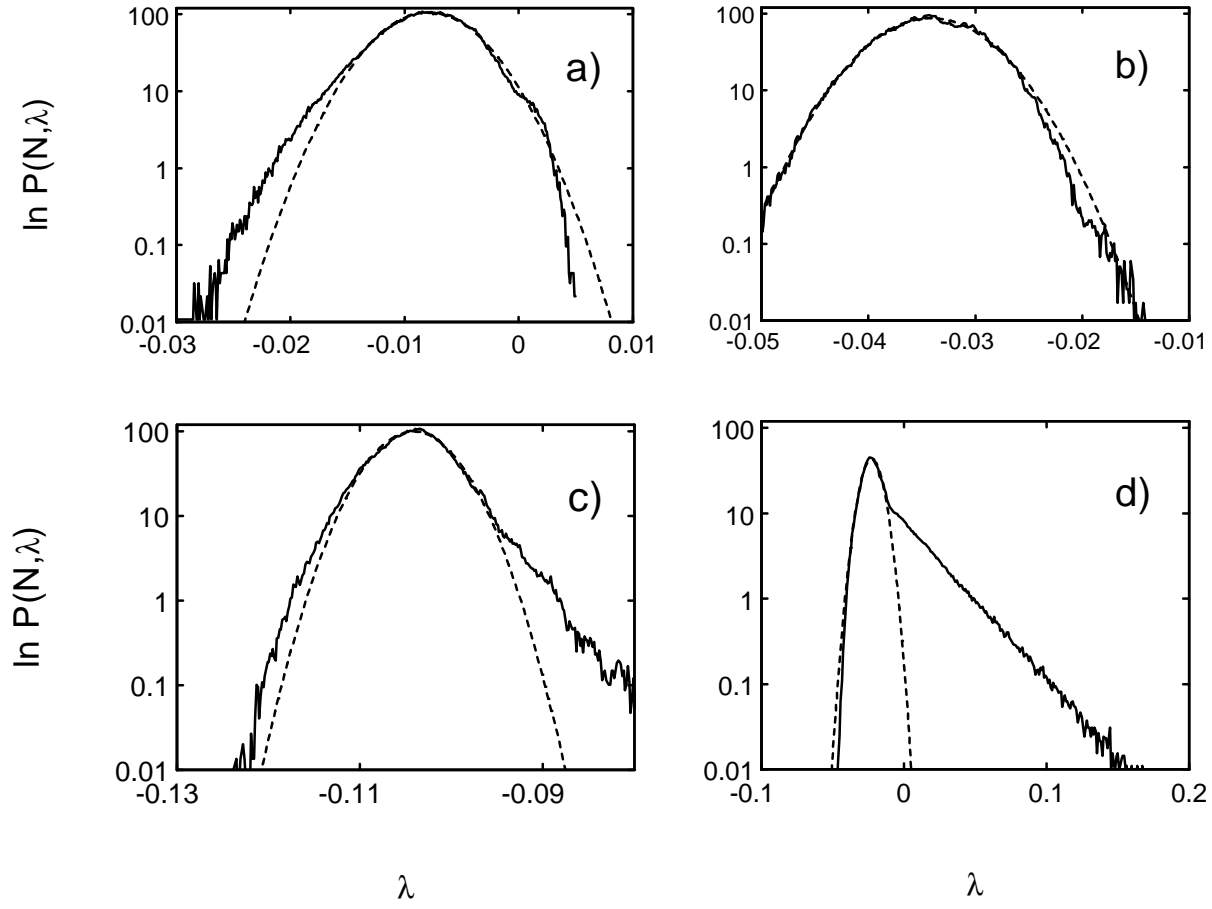


Fig. 3. As in Fig. 2, but for $N = 1000$. The parameters for the four systems are the same in Fig 2. (a)–(d). A gaussian is fit at the maximum of the density, and is shown here as a dotted line.

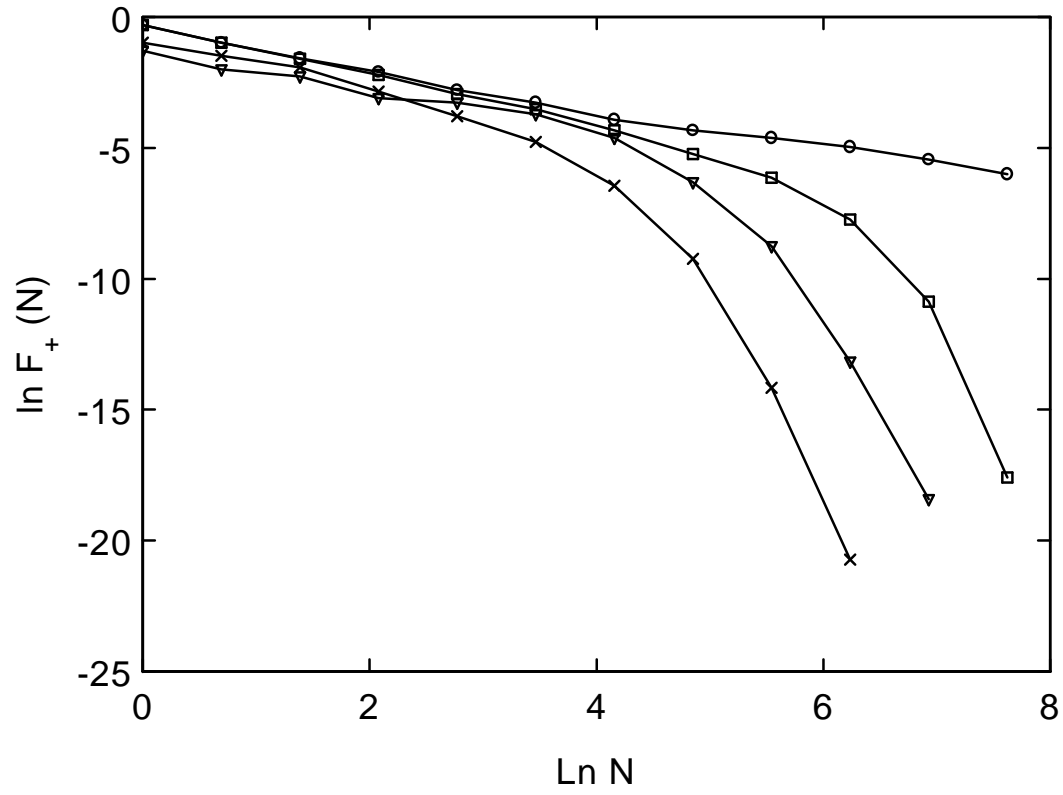


Fig. 4. Variation of $F_+(N)$ for the different SNAs, the parameters being identical to those in Figs. 2. and 3., the fractalized route (\square), the HH route (∇), the blowout bifurcation route (\times), and the intermittency route (\circ).

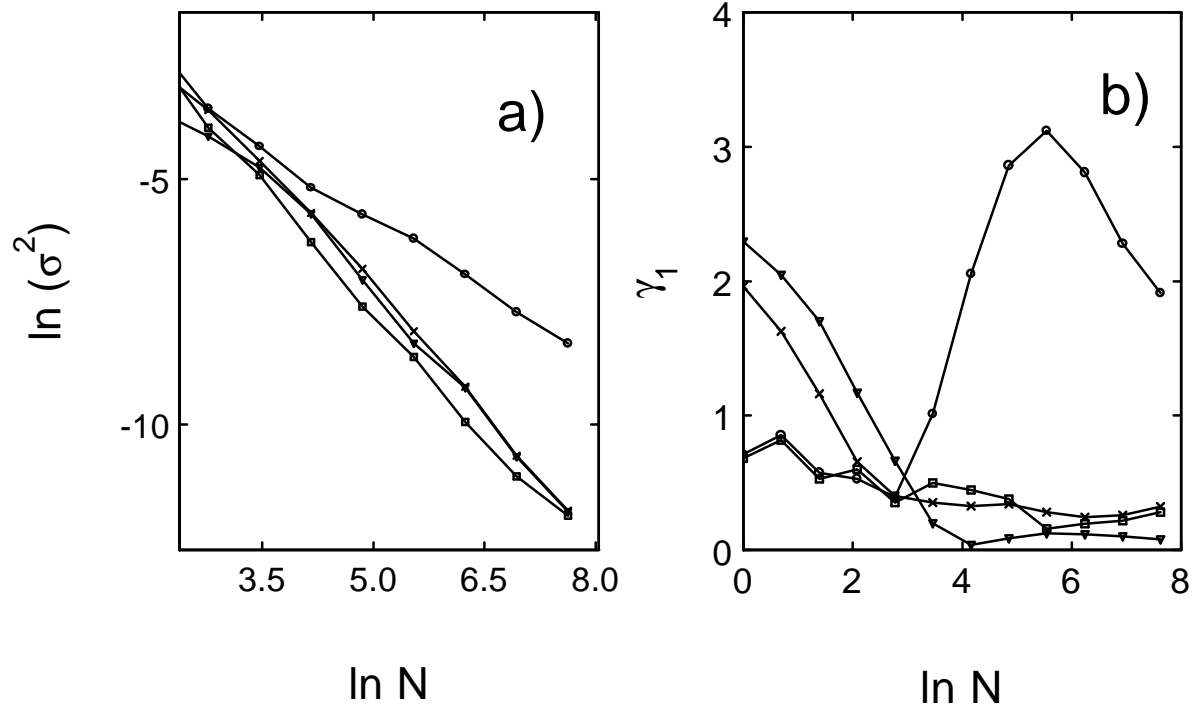


Fig. 5. Variation of the (a) variance, σ^2 and (b) the coefficient of skewness (γ_1) with N for the different SNAs. The symbols and parameter values are the same as in Fig. 4.